INTEGRABILITY + SUPERSYMMETRY + BOUNDARY: LIFE ON THE EDGE IS NOT SO DULL AFTER ALL!

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After a brief review of integrability, first in the absence and then in the presence of a boundary, I outline the construction of actions for the $N = 1$ and $N = 2$ boundary sine-Gordon models. The key point is to introduce Fermionic boundary degrees of freedom in the boundary actions.

1. Introduction

Quantum field theories (QFTs) with enhanced spacetime symmetries, such as integrability or supersymmetry, are attractive to theorists both as candidate models of real physical systems, and as toy models which can be probed more deeply than would otherwise be possible by exploiting their symmetries. Introducing a spatial boundary in such theories, whose effects can be physically important, poses a particular challenge to theorists, since boundary conditions generically break bulk spacetime symmetries.

Hence the fundamental question: to what extent can bulk spacetime symmetries be maintained in the presence of a spatial boundary? One expects that the "more" bulk symmetries there are, the harder it is to maintain such symmetries when a boundary is introduced. In particular, it was believed for some years that it is essentially impossible to maintain both integrability and supersymmetry in the presence of a boundary.

My main message here is that this belief is wrong: there **do** exist non-trivial integrable supersymmetric boundary QFTs. Although I address this question in the specific context of the sine-Gordon model, I expect that corresponding results can also be found for other models. This result may have

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applications in various areas, including condensed matter physics (in connection with impurity problems) and superstring theory. However, I have been motivated not so much by any particular application, but rather, by the two general convictions that systems with enhanced spacetime symmetries can be very interesting, and that boundary effects can be very important.

The remainder of this article is organized as follows. Sec. 2 briefly reviews some general features of integrability in the absence of boundaries, and considers as an example the sine-Gordon model [1]. Sec. 3 briefly reviews integrability in the presence of a boundary, focusing on the boundary sine-Gordon model [2]. Secs. 4 and 5 outline the construction of actions for the $N = 1$ and $N = 2$ boundary sine-Gordon models, respectively [3,4]. The key point is to introduce Fermionic boundary degrees of freedom in the boundary actions. Sec. 6 lists some interesting open problems, and points out related recent work on superconformal boundary Liouville models.

2. Integrability

In this Section, I very briefly review some general features of integrability in the absence of boundaries, and consider as an example the sine-Gordon model. See Zamolodchikov and Zamolodchikov [1] for a much more detailed review.

2.1. *Generalities*

A QFT is **integrable** if it has an infinite set of mutually commuting local integrals of motion (IMs). According to the Coleman-Mandula theorem [5], an integrable, Lorentz-invariant QFT in D spatial dimensions has a trivial S matrix, unless $D = 1$. Therefore, I henceforth restrict to 1 spatial dimension, with coordinate x. In this Section, I assume that there is no spatial boundary; i.e., the theory is defined on the line $-\infty < x < \infty$.

A trivial example of an integrable QFT is the theory of a free massive scalar field $\phi(x,t)$, with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \,. \tag{1}$$

Consider the following integrals of local densities [6,7]

$$\begin{aligned} I_{2n} &= \int_{-\infty}^{\infty} dx \ : \left[\frac{1}{2}(\partial_t\partial_x^n\phi)^2 + \frac{1}{2}(\partial_x^{n+1}\phi)^2 + \frac{1}{2}m^2(\partial_x^n\phi)^2\right] : \,, \\ I_{2n+1} &= \int_{-\infty}^{\infty} dx \ : \partial_t\phi \ \partial_x^{2n+1}\phi : \,, \qquad n = 0\,,1\,,2\,,\ldots \end{aligned} \tag{2}$$

Using the equation of motion $\partial_t^2 \phi = (\partial_x^2 - m^2)\phi$, one can check that these quantities are conserved $\frac{d}{dt} I_n = 0$, $n = 0\,, 1\,, 2\,, \ldots$, and are mutually commuting $[I_n\,, I_m] = 0$. Since I_0 is the energy and I_1 is the momentum, the quantities I_n are evidently higher-derivative generalizations of energy and momentum.

Since this is a free field theory, it is not surprising that it has infinitely many IMs. What is remarkable is that there do exist interacting field theories, such as the sine-Gordon model, that are integrable. An important consequence of integrability is that the multiparticle S matrix can be factorized into a product of two-particle S matrices, as if the scattering occurred by a series of **elastic** two-particle collisions, the movement of the particles in between being free. "Physicist proofs" of this fundamental result are given in Refs. [1] and [8]. An important consistency condition for this factorization is known as the Yang-Baxter Eq. By solving this equation, and imposing the constraints of crossing symmetry and unitarity, one can go a long way toward determining the **exact** two-particle (and hence, the full multiparticle) S matrix.

2.2. *The sine-Gordon model*

As an example, let us consider the so-called sine-Gordon model, which is among the first known and most-studied integrable QFTs. It is convenient to go to Euclidean space (x, y), with $z = x+iy$, $\bar{z} = x+iy$. The Lagrangian density is

$$\mathcal{L}_0 = 2\partial_z \varphi \partial_{\bar{z}} \varphi - \frac{m^2}{\beta^2} \cos(\beta\varphi)\,, \tag{3}$$

where $\varphi(z, \bar{z})$ is a real scalar field. Since φ is dimensionless (the number of spacetime dimensions being two), the coupling constant β is also dimensionless.

This model is known to be integrable at both the classical and quantum levels. To streamline the classical analysis, it is convenient to eliminate β from the field equation by rescaling the field, i.e., by defining $\phi = \beta\varphi$; and to fix the mass parameter by setting $m = 2$. This model has infinitely many conserved currents [6,7,9]

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1}\,, \qquad \partial_z \overline{T}_{s+1} = \partial_{\bar{z}} \overline{\Theta}_{s-1}\,, \qquad s = 1\,, 3\,, \ldots\,, \tag{4}$$

starting with

$$T_2 = (\partial_z \phi)^2 \,, \qquad \Theta_0 = -2\cos\phi \,,$$
$$T_4 = (\partial_z^2 \phi)^2 - \frac{1}{4}(\partial_z \phi)^4 \,, \qquad \Theta_2 = (\partial_z \phi)^2 \cos\phi \,, \tag{5}$$

and the corresponding barred quantities are obtained by complex conjugation. It follows from (4) that the local charges

$$P_s = \int_{-\infty}^{\infty} dx \ (T_{s+1} + \Theta_{s-1}) \,, \qquad \overline{P}_s = \int_{-\infty}^{\infty} dx \ (\overline{T}_{s+1} + \overline{\Theta}_{s-1}) \,, \tag{6}$$

are conserved

$$\frac{d}{dy} P_s = 0 = \frac{d}{dy} \overline{P}_s \,, \qquad s = 1\,, 3\,, \ldots \,. \tag{7}$$

The energy and momentum are given by $P_1 + \overline{P}_1$ and $P_1 - \overline{P}_1$, respectively. The charges with $s \ge 3$ are nontrivial – their existence proves the classical integrability of the model.

The classical spectrum includes solitons and antisolitons. Indeed, the classical potential $V(\varphi) = -\frac{m^2}{\beta^2}\cos(\beta\varphi)$ evidently has minima at $\varphi = 0$ (mod $\frac{2\pi}{\beta}$). There exist stable finite-energy solutions of the classical field equations which interpolate between neighboring minima. Defining the topological charge

$$T = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \ \partial_x \varphi = \frac{\beta}{2\pi} \left[\varphi(x = \infty\,, y) - \varphi(x = -\infty\,, y) \right] \,, \tag{8}$$

the solutions with $T = +1$ and $T = -1$ are called solitons and antisolitons, respectively. There are also solutions with $T = 0$ called breathers.

The quantum sine-Gordon model has particle-like states corresponding to these classical solutions, for $0 < \beta^2 < 8\pi$. The exact two-particle S matrix is given in [1].

3. Integrability in the presence of a boundary

Following [2], let us now consider what happens when a spatial boundary is introduced. That is, I consider an integrable QFT on the half line $-\infty < x \le 0$, which evidently has a boundary at $x = 0$. The first problem to be addressed is to determine the boundary conditions which preserve integrability. Another important problem is to determine the so-called boundary S matrix, which describes scattering off the boundary. Integrability implies that particles reflect elastically from the boundary,

and that the boundary S matrix obeys a boundary generalization [10] of the Yang-Baxter Eq.

Turning again to the example of the sine-Gordon model, let us consider the Lagrangian [2]

$$L = \int_{-\infty}^{0} dx \ \left(2\partial_z\phi\partial_{\bar{z}}\phi - 4\cos\phi\right) + B(\phi)\Big|_{x=0} . \tag{9}$$

The Lagrangian density is essentially (3) with the coupling constant scaled away and with $m = 2$. The boundary potential $B(\phi)$ does not change the bulk equation of motion, but does affect the boundary condition, which follows from the variational principle,

$$\left(\partial_x\phi + \frac{\partial B}{\partial\phi}\right)\Big|_{x=0} = 0 . \tag{10}$$

The question is: which $B(\phi)$ (if any) preserves integrability? Clearly, the corresponding boundary conditions must be compatible with some nontrivial IMs. Since the boundary breaks translational invariance, momentum-like quantities $\sim P_s - \overline{P}_s$ cannot be conserved. The only hope is for energy-like quantities $\sim P_s + \overline{P}_s$ to be conserved. Hence, consider the quantity

$$H_s = \int_{-\infty}^{0} dx \ \left[(T_{s+1} + \Theta_{s-1}) + (\overline{T}_{s+1} + \overline{\Theta}_{s-1})\right] . \tag{11}$$

Computing the "time" derivative,

$$\begin{aligned} \frac{d}{dy}H_s &= \int_{-\infty}^{0} dx \ \partial_y \left[\quad \right] = \int_{-\infty}^{0} dx \ i\partial_x \left(T_{s+1} - \Theta_{s-1} - \overline{T}_{s+1} + \overline{\Theta}_{s-1}\right) \\ &= i\left(T_{s+1} - \Theta_{s-1} - \overline{T}_{s+1} + \overline{\Theta}_{s-1}\right)\Big|_{x=0} \equiv i\frac{d}{dy}\Sigma_s , \end{aligned} \tag{12}$$

where the second equality on the first line follows from current conservation (4). Hence, if there exists a quantity Σ_s obeying (12), then $H_s - i\Sigma_s$ is an IM. That is, the boundary potential $B(\phi)$ should be chosen so that (considering the first nontrivial case, $s = 3$)

$$\left(T_4 - \Theta_2 - \overline{T}_4 + \overline{\Theta}_2\right)\Big|_{x=0} = \frac{d}{dy}\Sigma_3 . \tag{13}$$

Ghoshal and Zamolodchikov [2] solved the constraint (13) for the boundary potential, and obtained the result

$$B(\phi) = 2\alpha\cos\left(\frac{1}{2}(\phi - \phi_0)\right) , \tag{14}$$

where α and ϕ_0 are arbitrary boundary parameters. The model (9), (14) is known as the boundary sine-Gordon model. The corresponding boundary

condition (10), which reads $\left(\partial_x \phi - \alpha \sin(\frac{1}{2}(\phi - \phi_0))\right)\Big|_{x=0} = 0$, interpolates between Neumann ($\alpha = 0$) and Dirichlet ($\alpha \to \infty$) boundary conditions. Ghoshal and Zamolodchikov conjectured that this model is integrable at both the classical and quantum levels, and proposed the boundary S matrix for the solitons [2] and breathers [11]. Because there are (two) boundary parameters, the boundary S matrix exhibits a rich boundary boundstate structure [12,13].

4. Integrability and $N = 1$ supersymmetry in the presence of a boundary

The "bulk" sine-Gordon model (3) has a supersymmetric generalization, the so-called $N = 1$ sine-Gordon model [14]

$$\mathcal{L}_0 = 2\left(\partial_z\phi\partial_{\bar{z}}\phi - \bar{\psi}\partial_z\bar{\psi} + \psi\partial_{\bar{z}}\psi - 2\cos\phi - 2\bar{\psi}\psi\cos\frac{\phi}{2}\right) , \tag{15}$$

where ψ and $\bar{\psi}$ are components of a Majorana Fermion field. (Again, the dimensionless bulk coupling constant has been scaled away, and the mass parameter has been fixed to $m = 2$.) Indeed, this model has conserved supercurrents

$$\partial_{\bar{z}} T_{\frac{3}{2}} = \partial_z \Theta_{-\frac{1}{2}} \,, \qquad \partial_z \overline{T}_{\frac{3}{2}} = \partial_{\bar{z}} \overline{\Theta}_{-\frac{1}{2}} \,, \tag{16}$$

where

$$T_{\frac{3}{2}} = \partial_z\phi\psi \,, \qquad \Theta_{-\frac{1}{2}} = -2\bar{\psi}\sin\frac{\phi}{2} \,; \tag{17}$$

and corresponding conserved supersymmetry charges

$$P_{\frac{1}{2}} = \int_{-\infty}^{\infty} dx\ (T_{\frac{3}{2}} + \Theta_{-\frac{1}{2}}) \,, \qquad \overline{P}_{\frac{1}{2}} = \int_{-\infty}^{\infty} dx\ (\overline{T}_{\frac{3}{2}} + \overline{\Theta}_{-\frac{1}{2}}) \,. \tag{18}$$

Moreover, this model has an infinite set of local integrals of motion [15]

$$\frac{d}{dy} P_s = 0 = \frac{d}{dy} \overline{P}_s \,, \qquad s = 1\,, 3\,, \ldots \,, \tag{19}$$

(the corresponding currents for $s = 1\,, 3$ are generalizations of (5) with additional terms involving the Fermion field) and is therefore integrable. Bulk S matrices have been proposed for the solitons [16] and breathers [8,17].

One finally arrives at the question: are there boundary interactions which preserve both integrability and supersymmetry? This question was

first addressed by Inami, Odake and Zhang, who proposed the Lagrangian [18]

$$L = \int_{-\infty}^{0} dx\ \mathcal{L}_0 + B(\phi, \psi, \bar{\psi})\Big|_{x=0}\,, \tag{20}$$

where $\mathcal{L}_0$ is given by (15), and $B(\phi, \psi, \bar{\psi})$ is a boundary potential. Imposing both integrability (as in Eq. (13)) and supersymmetry, they found that the boundary potential is fixed up to a sign,

$$B(\phi, \psi, \bar{\psi}) = \pm \left(8 \cos \frac{\phi}{2} + \bar{\psi}\psi \right)\,. \tag{21}$$

That is, unlike the nonsupersymmetric ($N = 0$) boundary sine-Gordon model (9), (14), this model has **no** boundary parameters.

This no-go result (namely, that the combined constraints of integrability and supersymmetry do not allow any free parameters in the boundary action) seemed to me aesthetically unsatisfactory and even paradoxical [19]. Hence, I decided to revisit this problem [3]. My main idea was to introduce a Fermionic boundary degree of freedom in the boundary action. Indeed, the $N = 1$ solitons have an Ising-type RSOS degree of freedom [16], and Ghoshal and Zamolodchikov [2] introduced such a Fermionic boundary degree of freedom to describe the Ising model in a boundary magnetic field. Thus, instead of (20), I proposed the Lagrangian

$$L = \int_{-\infty}^{0} dx\ \mathcal{L}_0 + \left[\pm\bar{\psi}\psi + ia\frac{d}{dy}a - 2f(\phi)a(\psi \mp \bar{\psi}) + B(\phi) \right] \Big|_{x=0}\,, \tag{22}$$

where $\mathcal{L}_0$ is given by (15), a is a Hermitian Fermionic boundary degree of freedom which anticommutes with both ψ and $\bar{\psi}$, and $B(\phi)$, $f(\phi)$ are boundary potentials. Imposing both integrability (as in Eq. (13)) and supersymmetry ($\sim P_{\frac{1}{2}} \pm \overline{P}_{\frac{1}{2}}$), one finds that the boundary potentials are given by [3]

$$B(\phi) = 2\alpha \cos(\frac{1}{2}(\phi - \phi_0))\,, \qquad f(\phi) = \frac{\sqrt{C}}{2} \sin(\frac{1}{4}(\phi - D))\,, \tag{23}$$

where C, D are known functions of α, ϕ_0. That is, the $N = 1$ boundary sine-Gordon model (22), (23) has two arbitrary boundary parameters (α, ϕ_0) – the same as the $N = 0$ model (9), (14)! A boundary S matrix for the $N = 1$ boundary sine-Gordon model, which also depends on two boundary parameters, was subsequently proposed by Bajnok et *al.* [20].

5. Integrability and $N = 2$ supersymmetry in the presence of a boundary

Encouraged by these results, I then decided to consider the $N = 2$ case [4]. Indeed, the "bulk" sine-Gordon model (3) also has an $N = 2$ supersymmetric generalization [21]

$$\mathcal{L}_0 = \frac{1}{2}\big(- \partial_z\varphi^-\partial_{\bar{z}}\varphi^+ - \partial_{\bar{z}}\varphi^-\partial_z\varphi^+ + \bar{\psi}^-\partial_z\bar{\psi}^+ + \psi^-\partial_{\bar{z}}\psi^+ + \bar{\psi}^+\partial_z\bar{\psi}^- + \psi^+\partial_{\bar{z}}\psi^-\big) + g\cos\varphi^+\bar{\psi}^-\psi^- + g\cos\varphi^-\bar{\psi}^+\psi^+ + g^2\sin\varphi^+\sin\varphi^- \,, \tag{24}$$

where $\varphi^\pm$ form a complex scalar field; $\psi^\pm$ and $\bar{\psi}^\pm$ are the components of a complex Dirac Fermion field; g is the bulk mass parameter; and here $z = \frac{1}{2}(y + ix)$, $\bar{z} = \frac{1}{2}(y - ix)$. (Again, the dimensionless bulk coupling constant has been scaled away.)

This model has conserved supercurrents

$$\partial_{\bar{z}} T^\pm_{\frac{3}{2}} = \partial_z \Theta^\pm_{-\frac{1}{2}} \,, \qquad \partial_z \overline{T}^\pm_{\frac{3}{2}} = \partial_{\bar{z}} \overline{\Theta}^\pm_{-\frac{1}{2}} \,, \tag{25}$$

where

$$T^\pm_{\frac{3}{2}} = \partial_z\varphi^\pm\psi^\pm \,, \qquad \Theta^\pm_{-\frac{1}{2}} = g\bar{\psi}^\mp \sin\varphi^\pm \,; \tag{26}$$

and corresponding conserved supersymmetry charges

$$P^\pm_{\frac{1}{2}} = \int_{-\infty}^{\infty} dx\ (T^\pm_{\frac{3}{2}} - \Theta^\pm_{-\frac{1}{2}}) \,, \qquad \overline{P}^\pm_{\frac{1}{2}} = \int_{-\infty}^{\infty} dx\ (\overline{T}^\pm_{\frac{3}{2}} - \overline{\Theta}^\pm_{-\frac{1}{2}}) \,. \tag{27}$$

Moreover, this model has an infinite set of local integrals of motion

$$\frac{d}{dy} P_s = 0 = \frac{d}{dy} \overline{P}_s \,, \qquad s = 1\,, 3\,, \ldots \,, \tag{28}$$

and is therefore integrable. The bulk S matrix was proposed in [22].

This model can be formulated on the half-line most simply when the bulk mass vanishes ($g = 0$), in which case a suitable Lagrangian is [4] (see also [23])

$$L = \int_{-\infty}^{0} dx\ \mathcal{L}_0 + \Big[- \frac{i}{2}(\bar{\psi}^+\psi^- + \bar{\psi}^-\psi^+) - \frac{1}{2}a^-\frac{d}{dy}a^+ - B(\varphi^+\,, \varphi^-) + \frac{1}{2}f^+(\varphi^+)a^+(\psi^- + \bar{\psi}^-) + \frac{1}{2}f^-(\varphi^-)a^-(\psi^+ + \bar{\psi}^+)\Big]\Big|_{x=0} \,, \tag{29}$$

where $\mathcal{L}_0$ is given by (24), $a^\pm$ are Fermionic boundary degrees of freedom which anticommute with $\psi^\pm$ and $\bar{\psi}^\pm$, and $B(\varphi^+\,, \varphi^-)$, $f^\pm(\varphi^\pm)$ are

boundary potentials. Imposing both integrability (as in Eq. (13)) and supersymmetry ($\sim P^{\pm}_{\frac{1}{2}} + \overline{P}^{\pm}_{\frac{1}{2}}$), one finds that the boundary potentials are given by [4]

$$
\begin{aligned}
B(\varphi^{+}, \varphi^{-}) &= \alpha \cos(\frac{1}{2}(\varphi^{+} - \varphi_0^{+})) \cos(\frac{1}{2}(\varphi^{-} - \varphi_0^{-}))\,, \\
f^{\pm}(\varphi^{\pm}) &= \frac{\sqrt{\alpha}}{2} \sin(\frac{1}{2}(\varphi^{\pm} - \varphi_0^{\pm}))\,.
\end{aligned}
\tag{30}
$$

Hence, there are three (!) boundary parameters α, $\varphi_0^{\pm}$. For the bulk massive case ($g \neq 0$), the boundary action has more terms, and I have performed an analysis only up to first order in g. The boundary S matrix for this model has been discussed by Baseilhac and Koizumi [24]. (See also [23,25].)

6. Outlook

Already for the the nonsupersymmetric ($N = 0$) boundary sine-Gordon model, there are many interesting questions that remain unanswered, such as its relation to ϕ_{13}-perturbed boundary minimal conformal field theories (CFTs). In the bulk case, it is known [26] that the S matrices of the perturbed minimal models are restrictions of the sine-Gordon S matrix. One would like to know if something similar happens in the boundary case.

For the minimal models, the possible conformal boundary conditions (CBCs) have been classified by Cardy [27]. (A CBC is characterized in part by its boundary entropy [28], similar to the way that a bulk CFT is characterized by its central charge.) The boundary S matrix of a perturbed CFT describes the boundary "flow" from one CBC to another.

Such issues (and more!) can eventually also be addressed for the $N = 1$ and $N = 2$ boundary sine-Gordon models which have been discussed here.

This work has also recently led to progress in formulating superconformal boundary Liouville models. As is well known, the Liouville model is closely related to the sine-Gordon model. It is conformal invariant, not just integrable. For the $N = 1$ and $N = 2$ boundary Liouville models, the same Ansätze (22), (29) give one-parameter families of boundary actions which are $N = 1$ [29,30] and $N = 2$ [31] superconformal invariant, respectively.

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References

1. A.B. Zamolodchikov and Al.B. Zamolodchikov, *Ann. Phys.* **120**, 253 (1979).

2. S. Ghoshal and A.B. Zamolodchikov, *Int. J. Mod. Phys.* **A9**, 3841 (1994).
3. R.I. Nepomechie, *Phys. Lett.* **B509**, 183 (2001).
4. R.I. Nepomechie, *Phys. Lett.* **B516**, 376 (2001).
5. S.R. Coleman and J. Mandula, *Phys. Rev.* **159**, 1251 (1967).
6. P.P. Kulish, "Conservation laws for the sine-Gordon equation," Serpukhov preprint (1974); *Theor. Math. Phys.* **26**, 132 (1976).
7. V.E. Korepin and L.D. Faddeev, *Theor. Math. Phys.* **25**, 1039 (1975).
8. R. Shankar and E. Witten, *Phys. Rev.* **D17**, 2134 (1978).
9. L.D. Faddeev and L.A. Takhtajan, *Theor. Math. Phys.* **21**, 1046 (1975).
10. I.V. Cherednik, *Theor. Math. Phys.* **61**, 977 (1984).
11. S. Ghoshal, *Int. J. Mod. Phys.* **A9**, 4801 (1994).
12. P. Mattsson and P. Dorey, *J. Phys.* **A33**, 9065 (2000).
13. Z. Bajnok, L. Palla, G. Takacs and G.Z. Toth, *Nucl. Phys.* **B622**, 548 (2002).
14. P. Di Vecchia and S. Ferrara, *Nucl. Phys.* **B130**, 93 (1977); J. Hruby, *Nucl. Phys.* **B131**, 275 (1977).
15. S. Ferrara, L. Girardello and S. Sciuto, *Phys. Lett.* **B76**, 303 (1978); L. Girardello and S. Sciuto, *Phys. Lett.* **B77**, 267 (1978); R. Sasaki and I. Yamanaka, *Prog. Theor. Phys.* **79**, 1167 (1988).
16. C. Ahn, D. Bernard and A. LeClair, *Nucl. Phys.* **B346**, 409 (1990).
17. C. Ahn, *Nucl. Phys.* **B354**, 57 (1991).
18. T. Inami, S. Odake and Y-Z Zhang, *Phys. Lett.* **B359**, 118 (1995).
19. C. Ahn and W.M. Koo, *J. Phys.* **A29**, 5845 (1996); *Nucl. Phys.* **B482**, 675 (1996); M. Moriconi and K. Schoutens, *Nucl. Phys.* **B487**, 756 (1997); A. Chenaghlou and E. Corrigan, *Int. J. Mod. Phys.* **A15**, 4417 (2000); M. Ablikim and E. Corrigan, *Int. J. Mod. Phys.* **A16**, 625 (2001); C. Ahn and R.I. Nepomechie, *Nucl. Phys.* **B594**, 660 (2001).
20. Z. Bajnok, L. Palla and G. Takacs, *Nucl. Phys.* **B644**, 509 (2002).
21. K. Kobayashi and T. Uematsu, *Phys. Lett.* **B264**, 107 (1991).
22. K. Kobayashi and T. Uematsu, *Phys. Lett.* **B275**, 361 (1992).
23. N.P. Warner, *Nucl. Phys.* **B450**, 663 (1995).
24. P. Baseilhac and K. Koizumi, *Nucl. Phys.* **B669**, 417 (2003).
25. R.I. Nepomechie, *Phys. Lett.* **B516**, 161 (2001).
26. A. Leclair, *Phys. Lett.* **B230**, 103 (1989); D. Bernard and A. LeClair, *Nucl. Phys.* **B340**, 721 (1990); N. Reshetikhin and F. Smirnov, *Commun. Math. Phys.* **131**, 157 (1990).
27. J.L. Cardy, *Nucl. Phys.* **B324**, 581 (1989).
28. I. Affleck and A.W.W. Ludwig, *Phys. Rev. Lett.* **67**, 161 (1991).
29. T. Fukuda and K. Hosomichi, *Nucl. Phys.* **B635**, 215 (2002).
30. C. Ahn, C. Rim and M. Stanishkov, *Nucl. Phys.* **B636**, 497 (2002).
31. C. Ahn and M. Yamamoto, *Phys. Rev.* **D69**, 026007 (2004).